ON JAMES'S PAPER "SEPARABLE CONJUGATE SPACES"

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ABSTRACT

For every separable Banach space X there is a Banach space Y with a separable dual such that $Y \oplus X^* \approx Y^{**}$. There is also a separable space Z so that Z^{**}/JZ is isomorphic to X.

The present note is a direct outgrowth of James's paper [4]. The purpose of the note is twofold: (1) to prove a more general form of James's result, and (2) to give a simpler exposition of James's proof (but not a really different proof). Since the spaces constructed in [4] are among the most striking examples (or better, counter-examples) in Banach space theory it seemed to us worthwhile to present this somewhat simpler and more general version of the paper [4].

THEOREM. *Let X be a separable Banach space. Then there exists a separable Banach space Y so that*

i) *Y has a monotone shrinking basis.*

ii) *There is a quotient map ϕ from Y^* onto X.*

iii) *$Y^{**} = JY \oplus \phi^* X^*$ (and hence $Y^{**} \approx Y \oplus X^*$).*

JY denotes the canonical embedding of Y in Y^{**}. For the basic definitions and results concerning Schauder bases we refer to [2] pp. 67–72. In [4] James proves essentially the same result under the assumption that X^* is separable and has a boundedly complete basis.

PROOF. Let $\{x_i\}_{i=1}^{\infty}$ be a dense sequence on the boundary of the unit cell of X. Let Y^* consist of all the sequences $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ of scalars such that

$$\|\alpha\| = \sup \Big(\sum_{j=1}^{k} \Big\| \sum_{i=n_{j-1}+1}^{n_j} \alpha_i x_i \Big\|^2 \Big)^{\frac{1}{2}} < \infty$$

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where the sup is taken over all $k = 1,2,\cdots$ and all finite sequences of integers $0 = n_0 < n_1 < n_2 \cdots < n_k$.

Let $e_k = \{\delta_k^i\}_{i=1}^{\infty}$ be the k-th unit vector in Y^*. It is clear from the definition that the $\{e_k\}_{k=1}^{\infty}$ form a monotone boundedly complete basis of Y^* and that $\|e_k\| = 1$ for all k. Hence, as well known (cf. [2]), Y^* is isometric to the conjugate of a Banach space Y in such a manner that $JY \subset Y^{**}$ is the closed linear span of biorthogonal functionals $\{f_k\}_{k=1}^{\infty}$ to the basis $\{e_k\}_{k=1}^{\infty}$. It is also clear that if $\alpha = \{\alpha_i\}_{i=1}^{\infty} \in Y^*$ then $\sum_{i=1}^{\infty} \alpha_i x_i$ converges in norm and $\phi: \alpha \to \sum_{i=1}^{\infty} \alpha_i x_i$ is an operator of norm ≤ 1 from Y^* into X. Since $\phi(e_k) = x_k$ it follows that ϕ is a quotient map. Hence $\phi^*: X^* \to Y^{**}$ is an isometry into. If $u = \sum_{i=1}^{\infty} \beta_i f_i$ is an element of JY and $x^* \in X^*$ then $\lim_i \beta_i = 0$ and $(\phi^*(x^*) + u)(e_i) = x^*(x_i) + \beta_i$ and thus

$$\|\phi^*(x^*) + u\| \geq \sup_i |x^*(x_i) + \beta_i| \geq \overline{\lim_i} |x^*(x_i)| = \|x^*\| = \|\phi^*(x^*)\|.$$

Hence $JY \cap \phi^*X^* = (0)$ and the map from $JY + \phi^*X^*$ onto ϕ^*X^* which maps JY into 0 is a projection of norm 1. The only fact which remains to be proved (and this is non trivial) is that $JY \oplus \phi^*X^*$ exhausts all of Y^*. It is clearly enough to show that if $y^{**} \in Y^{**}$ with $\|y^{**}\| = 1$ then its distance from $JY \oplus \phi^*X^*$ is $\leq 7/8$.

It follows from the definition of the norm in Y^* and the standard separation theorem that the norm closed convex hull of the set A (defined below) is the closed unit cell of $JY \subset Y^{**}$. We say that $u \in A$ if there are integers k and $0 = n_0 < n_1 < n_2 < \cdots < n_k$ and elements $\{x_j^*\}_{j=1}^{k}$ in X^* such that

$$(1) \qquad u = \sum_{j=1}^{k} \sum_{i=n_{j-1}+1}^{n_j} x_j^*(x_i) f, \qquad \sum_{j=1}^{k} \|x_j^*\|^2 \leq 1.$$

The integers n_j appearing in (1) are called the *division points* of u. (The representation (1) is not unique in general. This, however, need not bother us: we simply choose for every u appearing below one fixed representation.)

Let $y^{**} \in Y^{**}$ be of norm 1. There exists a sequence

$$(2) \qquad v_m = \sum_{l=1}^{\gamma_m} \lambda_{l,m} u_{l,m}, \qquad \lambda_{l,m} \geq 0, \qquad \sum_{l=1}^{\gamma_m} \lambda_{l,m} = 1, \qquad u_{l,m} \in A$$

which converges w^* to y^{**} as $m \to \infty$. We consider now separately the following two cases.

Case a. There exists an integer i_0 such that for every $i > i_0$, $\overline{\lim}_m \sum' \lambda_{l,m} \geq 1/8$ where in $\sum'$ we sum only over those indices l for which $u_{l,m}$ does not have a division point between i_0 and i. (The set of indices which enter into Σ' depends of course on i_0 and i.)

Case b. There exists no such i_0.

Suppose case a holds. Then there exists a subsequence of $\{v_m\}_{m=1}^{\infty}$ (which, for sake of simplicity, we continue to call $\{v_m\}_{m=1}^{\infty}$) such that $v_m = v_m' + v_m''$ with $v_m' = \Sigma'\lambda_{l,m}u_{l,m}$ where each $u_{l,m}$ has no point of division between i_0 and some i_m, $\Sigma'\lambda_{l,m} \geqq 1/8$ and $\lim_m i_m = \infty$. Clearly $\|v_m''\| \leqq 7/8$. Without loss of generality we may assume (pass to a subsequence if necessary) that $w^* \lim v_m''$ exists and is equal to z^{**} say. Then $\|z^{**}\| \leqq 7/8$ and $w^* \lim v_m' = y^{**} - z^{**}$. By our assumption on the $u_{l,m}$ which enter into v_m' there is a $t_m^* \in X^*$ with $\|t_m^*\| \leqq \Sigma'\lambda_{l,m} \leqq 1$ such that $v_m'(e_i) = t_m^*(x_i)$ for $i_0 \leqq i \leqq i_m$. Again, there is no loss of generality to assume that $t^* = w^* \lim t_m^*$ exists. For $i \geqq i_0$ we have $(y^{**} - z^{**})(e_i) = t^*(x_i) = \phi^* t^*(e_i)$. In other words $y^{**} - z^{**}$ differs from $\phi^* t^*$ by an element in the span of $\{f_i\}_{i=1}^{i_0}$ and thus in JY. Hence the distance of y^{**} from $JY \oplus \phi^* X^*$ is $\leqq \|z^{**}\| \leqq 7/8$.

Before treating case b we make a trivial remark. Assume that α_i, β_i, and λ_i are non negative numbers with $\sum_i \lambda_i \leqq 1$ and $\alpha_i^2 + \beta_i^2 \leqq 1$ for all i. Then

$$(3) \qquad (\sum_i \alpha_i \lambda_i)^2 + (\sum_i \beta_i \lambda_i)^2 \leqq (\sum_i \lambda_i)(\sum_i \lambda_i(\alpha_i^2 + \beta_i^2)) \leqq 1.$$

Assume now that case b holds. Choose i_0 so that

$$(4) \qquad \Big\| \sum_{i=1}^{i_0} y^{**}(e_i) f_i \Big\| \geqq 7/8.$$

(This is possible since $1 = \|y^{**}\| = \lim_{m \to \infty} \| \sum_{i=1}^{m} y^{**}(e_i) f_i \|$.) Since we are in case b there is an $i_1 > i_0$ such that for all sufficiently large m, $\Sigma''\lambda_{l,m} \geqq 7/8$ where the sum Σ'' is taken over all those indices such that $u_{l,m}$ has a division point between i_0 and i_1. For every l entering into Σ'' put $u_{l,m} = t_{l,m} + z_{l,m}$ where $t_{l,m} \in JY$ is defined by $t_{l,m}(e_i) = u_{l,m}(e_i)$ for $i \leqq i(l,m)$, $t_{l,m}(e_i) = 0$ for $i > i(l,m)$ and $i(l,m)$ is the first division point of $u_{l,m}$ after i_0 (by definition, $i(l,m) \leqq i_1$). It follows from (1) that

$$(5) \qquad 1 \geqq \|u_{l,m}\|^2 \geqq \|t_{l,m}\|^2 + \|z_{l,m}\|^2.$$

Let $t_m = \Sigma''\lambda_{l,m}t_{l,m}$, $z_m = \Sigma''\lambda_{l,m}z_{l,m}$ and $v_m' = v_m - (t_m + z_m)$. Clearly $\|v_m'\| \leqq 1 - \Sigma''\lambda_{l,m} \leqq 1/8$ and $z_m(e_i) = 0$ for $i \leqq i_0$. Without loss of generality we may assume that $v^{**} = w^* \lim v_m'$, $t^{**} = w^* \lim t_m$ and $z^{**} = w^* \lim z_m$ exist. Clearly $\|v^{**}\| \leqq 1/8$, $y^{**} = v^{**} + t^{**} + z^{**}$ and $z^{**}(e_i) = 0$ for $i \leqq i_0$. Hence by (4)

$$\|t^{**}\| \geqq \| \sum_{i=1}^{i_0} t^{**}(e_i)f_i\| = \| \sum_{i=1}^{i_0} (y^{**}-v^{**})(e_i)f_i\| \geqq 7/8 - \|v^{**}\| \geqq 3/4.$$

It follows that $\underline{\lim}_m \Sigma'' \lambda_{l,m}\|t_{l,m}\| \geqq 3/4$ and hence by (3) and (5)

$$\|z^{**}\| \leqq \overline{\lim}_m \Sigma'' \lambda_{l,m}\|z_{l,m}\| \leqq (1-(3/4)^2)^{\frac{1}{2}} = \sqrt{7}/4.$$

Since t^{**} is in the span of $\{f_i\}_{i=1}^{i_1}$ and thus in JY it follows that the distance of y^{**} from $JY \oplus \phi^* X^*$ is at most $\|v^{**}\| + \|z^{**}\| \leqq 1/8 + \sqrt{7}/4 < 7/8$, as desired.

REMARKS. 1) It is easy to verify directly that the w^* closure of A (and hence all the extreme points of the unit cell of Y^{**}) belong to $JY \oplus \phi^* X^*$. In case X^* is separable and we know a priori that Y^{**} is separable this would already imply (by [1]) that $Y^{**} = JY \oplus \phi^* X^*$. It would be nice if one could prove in general part (iii) of the theorem by using extreme points.

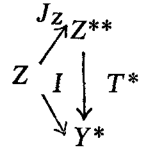
2) Define inductively the separable Banach spaces $\{X_n\}_{n=0}^{\infty}$ by $X_0 = c_0$ and $X_n^{**} \approx X_{n-1}^{*} \oplus X_n$. Then clearly the $(n+1)$th conjugate of X_n is separable while the $(n+2)$th conjugate is no longer separable. The construction of such $\{X_n\}_{n=1}^{\infty}$ was the main purpose of James in [4]. We are convinced that the spaces constructed by James (or more generally in the theorem above) will be useful as counterexamples also in other contexts. We state here four additional consequences of the theorem.

COROLLARY 1. *For every separable Banach space X there is a separable Banach space Z such that Z^{**}/JZ is isomorphic to X.*

PROOF. Let Z be the kernel of the quotient map ϕ of part (ii) of the theorem. More explicitly: Z consists of all sequences $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ such that

$$\|\alpha\| = \sup \left(\sum_{j=1}^{k} \| \sum_{i=n_{j-1}+1}^{n_j} \alpha_i x_i\|^2\right)^{\frac{1}{2}} < \infty \text{ and } \sum_{i=1}^{\infty} \alpha_i x_i = 0.$$

By standard facts concerning duality, Z^* is isometric to $Y^{**}/\phi^* X^*$ and hence, by part (iii), is isomorphic to Y. More explicitly; let $I: Z \to Y^*$ be the identity map and $J_Y: Y \to Y^{**}$ be the canonical embedding. Then $T = I^* J_Y$ is an isomorphism from Y onto Z^*. Let $J_Z: Z \to Z^{**}$ be the canonical embedding. The diagram



commutes. Indeed for every $z \in Z$ and $y \in Y$

$$T^* J_Z z(y) = J_Z z(Ty) = Ty(z) = I^* J_Y y(z) = J_Y y(Iz) = Iz(y).$$

Hence $Z^{**}/J_z Z$ is isomorphic to Y^*/IZ which in turn is isometric to X.

REMARK. $\|T\| \leqq 1$ and $\|T^{-1}\| \leqq 2$ and thus the distance coefficient $d(X, Z^{**}/JZ)$ is at most 2.

COROLLARY 2. *Let X be a separable infinite-dimensional Banach space. Then there is a monotone norm on the space of sequences of scalars which are eventually* 0 *so that X is isomorphic to the quotient B/C where*

$$B = \{\lambda = (\lambda_1 \lambda_2, \cdots, \lambda_n, \cdots);\ \|\lambda\| = \sup_n \|(\lambda_1, \lambda_2, \cdots, \lambda_n, 0\ 0\ \cdots)\| < \infty\}$$
$$C = \{\lambda = (\lambda_1 \lambda_2, \cdots, \lambda_n, \cdots);\ \lim_{m,n \to \infty} \|(0, 0, \cdots, 0, \lambda_m, \cdots, \lambda_n, 0, 0 \cdots,)\| = 0\}.$$

PROOF. This is an immediate consequence of Corollary 1 and the result of [5] which ensures that the space Z of Corollary 1 has a shrinking basis (since Z^* is known to have a Schauder basis).

The remaining corollaries are concerned with the approximation property. A Banach space X is said to have the approximation property if for every compact $K \subset X$ there is a bounded linear operator T on X such that dim $TX < \infty$ and $\|Tx - x\| \leqq 1$ for $x \in K$ (cf. [3]) Grothendieck proved in [3] that if X^* has the approximation property the same is true for X. In [5] there are some other results which show that if X^* behaves "well" the same is true for X. It is possible to go in the other direction i.e. pass from X to X^*? The next corollary shows that the answer to this question is negative (unless all Banach spaces turn out to behave "well").

COROLLARY 3. *If there is a Banach space which does not have the approximation property then there is a separable Banach space W which has a Schauder basis but whose dual W^* does not have the approximation property.*

PROOF. It is clear from the definition and the Hahn Banach theorem that if there is a Banach space which does not have the approximation property then there is also a separable X which fails to have the approximation property. Let Y be the space constructed in the Theorem corresponding to this space X. Let $W = Y^*$. By part (i) of the Theorem W has a Schauder basis. Since X^* is isomorphic to a complemented subspace of W^* (by (iii)) and X^* does not have the approximation property (by the result of [3] mentioned above) it follows that W^* does not have the approximation property.

REMARK. As shown by Grothendieck in [3] we may assume even that X is a subspace of c_0 (if there exists at all a space which fails to have the approximation property). Hence the space W may be constructed so that in addition W^* is separable.

COROLLARY 4. *A separable conjugate space has the approximation property if and only if it is a complemented subspace of a conjugate space with a Schauder basis.*

PROOF. The "if" part is trivial. To prove the "only if" part, let X be a Banach space such that X^* is separable and has the approximation property. Construct the space Y of the Theorem. The space Y^{**} is separable and by (i), (iii) and our assumption on X^*, Y^{**} has the approximation property. Since Y^* has a Schauder basis it follows from the results of [5] that Y^{**} has a Schauder basis. (We use here the fact which is, as pointed out to us by H.P. Rosenthal, contained implicitly in [3] that a separable conjugate space which has the approximation property has also the bounded approximation property in the sense of [5].) Since X^* is isometric to a complemented subspace of Y^{**}, the proof is complete.

REMARK. A result of a similar nature is proved in [6]: Every separable space (not necessarily conjugate space) which has the bounded approximation property is a complemented subspace of a space with a Schauder decomposition into finite-dimensional subspaces. Also [5] has some results in this direction.

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